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The $\mathscr{L}^p$ Space of a Positive Definite Matrix of Measures and Density of Matrix Polynomials in $\mathscr{L}^1$ *

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In this paper we study the space $\mathscr{L}^p(\mu)$, $1 \leqslant p \leqslant +\infty$, μ being a positive definite matrix of measures. We prove that the set of all positive definite matrices of measures having the same moments as those of μ is compact in the vague topology, and we give a density result for $\mathscr{L}^1(\mu)$ which is an extension to the matrix case of the classical result for scalar polynomials and positive measures due to Naimark.

1. INTRODUCTION

For a fixed non-negative integer N, we consider $N \times N$ positive definite matrices of measures μ; i.e., for any Borel set A, the numerical matrix $\mu(A)$ is positive semidefinite (hence μ is Hermitian). Associated to every μ, the space $\mathscr{L}^p(\mu)$, $1 \leqslant p \leqslant \infty$, is defined as follows:

Taking into account the inequality $\theta \leqslant \mu \leqslant (\tau\mu)\, I$ (here and in the rest of this paper, we write θ for the null matrix of size $N \times N$), we have that each measure $\mu_{i,j}$ is absolutely continuous with respect to the trace measure $\tau\mu$ (for matrix $A = (a_{i,j})_{1 \leqslant i, j \leqslant N}$, the trace is defined by $\tau A = \sum_{i=1}^{N} a_{i,i}$). Hence, the Radon–Nikodym derivatives $m_{i,j} = d\mu_{i,j}/d\tau\mu$ are well defined except for a set of null measures for the trace. The matrix of functions

$$M = (m_{i,j})_{i,j=1}^{N} = \left(\frac{d\mu_{i,j}}{d\tau\mu} \right)_{1 \leqslant i, j \leqslant N},$$

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which is positive semidefinite, is called a derivative of μ with respect to its trace. M is formed by measurable functions integrable with respect to $\tau\mu$, and for any Borel set A the equality

$$\int_A M(t)\, d\tau\mu(t) = \mu(A)$$

holds.

The inequality $\theta \leqslant M(t) \leqslant I$, which holds for t almost everywhere $(\tau\mu)$, is easily deduced from $\theta \leqslant \mu \leqslant (\tau\mu)\, I$.

We suppose the matrix $M(t)$ diagonalized as $M(t) = Q(t)\, \Lambda(t)\, Q^*(t)$, where $Q(t)$ is a unitary matrix and $\Lambda(t)$ is the diagonal matrix obtained by putting on its diagonal the eigenvalues $\lambda_1(t), ..., \lambda_N(t)$ in increasing order. We call $P(t)$ the matrix of the projection operator from $\mathbb{C}^N$ on the image of $M(t)$. $P(t)$ is obtained by putting $P(t) = Q(t)\, \tilde{\Lambda}(t)\, Q^*(t)$, where $\tilde{\Lambda}(t)$ is the matrix obtained from $\Lambda(t)$ by substituting its non-zero entries by 1.

Given p, $1 \leqslant p < \infty$, and n a fixed natural number, we define the space $\mathscr{L}_n^p(\mu)$ as the set of $n \times N$ matrix functions $f\colon \mathbb{R} \to M_{n \times N}(\mathbb{C})$ such that $\tau(f(t)\, M(t)^{2/p}\, f(t)^*)^{1/2} \in L^p(\tau\mu)$, and we define

$$\|f\|_{p,\mu} = \|\tau(f(t)\, M(t)^{2/p}\, f(t)^*)^{1/2}\|_{p,\tau\mu} = \left(\int_{\mathbb{R}} \tau(f(t)\, M(t)^{2/p}\, f(t)^*)^{p/2}\, d\tau\mu\right)^{1/p},$$

and if $p = \infty$, we say that $f \in \mathscr{L}_n^\infty(\mu)$ if $\tau(f(t)\, P(t)\, f(t)^*)^{1/2} \in L^\infty(\tau\mu)$ and then we define

$$\|f\|_{\infty,\mu} = \|\tau(f(t)\, P(t)\, f(t)^*)^{1/2}\|_{\infty,\tau\mu}.$$

In both cases, we identify $f(t)$ with $g(t)$ if $(f(t) - g(t))\, M(t) = \theta$.

These spaces have already been defined and studied, for $p = 2$ in [3, 13, 19], and for $1 \leqslant p \leqslant \infty$ in [14–16] in the more general context of operator valued functions. To make this paper more complete and self-contained we devote Section 2 to the study of the spaces $\mathscr{L}^p(\mu)$, with matrix values, and its properties. We will prove that they are Banach spaces and that the dual of $\mathscr{L}^p(\mu)$ is $\mathscr{L}^q(\mu)$, where $1 \leqslant p < +\infty$ and q are conjugate exponents (for $p = 2$, we have that $\mathscr{L}^2(\mu)$ is a Hilbert space). The proofs provided here for these facts are new and easier than those in [14] (or [3, 19] for $p = 2$).

These spaces appear in connection with the theory of orthogonal matrix polynomials (matrix polynomials which are orthogonal with respect to a positive definite matrix of measures) and they are of great interest because they provide a natural environment for approximating matrix functions using matrix polynomials. During the past few years, some important results in the theory of orthogonal polynomials have been extended to orthogonal matrix polynomials with the consequence that this theory of orthogonal matrix polynomials is receiving an increasing amount interest (see [4–10, 12, 20]).

In Section 3 of this paper we prove some results concerning topological properties of the space of positive definite matrices of measures: Let μ be a positive definite matrix of measures $\mu = (\mu_{i,j})_{i,j=1,\dots,N}$ satisfying $\int_{\mathbb{R}} t^n \, d|\mu_{i,j}(t)| < \infty$ for any $1 \leqslant i, j \leqslant N$ and for any n natural. The sequence $(\int_{\mathbb{R}} t^n \, d\mu(t))_n$ will be called the sequence of moments of μ. We can define the set $[\mu]$ of positive definite matrices of measures having the same sequence of moments as those of μ. As in the scalar case, the positive definite matrix of measures μ is called determinante if $[\mu]$ is a singleton and indeterminante if $[\mu]$ consists of more than one positive definite matrix of measures. We will prove that $[\mu]$ is again compact for the vague topology. Some related results can be found in [11].

In the scalar case, the characterization of the measures μ for which $\mathbb{C}[x]$ is dense in $L^1(\mu)$ was answered completely long ago by Naimark ([17]): $\mathbb{C}[x]$ is dense in $L^1(\mu)$ if and only if μ is an extreme point of the convex set $[\mu]$. The case $p = 2$ was also solved long ago by Riesz ([18]). More recently, other density results for $p > 2$ were obtained by Berg–Christensen ([2]), and Sodin ([21]). To complete this paper, we prove in Section 4 a partial extension of Naimark's result:

THEOREM 1. *Given a positive definite matrix of measures $\mu = M(t)\, d\tau\mu$, we call $0 \leqslant \lambda_1(t) \leqslant \cdots \leqslant \lambda_N(t) \leqslant 1$ to the eigenvalues of the matrix $M(t)$ ordered in increasing order. If there exists $\varepsilon > 0$ such that $\lambda_1(t) \geqslant \varepsilon$, for every t in the support of μ, then the following statements are equivalent*:

(1) *The polynomials are dense in the space $\mathscr{L}^1(\mu)$.*

(2) *μ is an extremal measure of the set $[\mu]$.*

The existence of extremal points in the convex set $[\mu]$ follows from the compactness of $[\mu]$.

2. THE SPACES $\mathscr{L}^p$: DUALITY

From the definition of the space $\mathscr{L}_n^\infty(\mu)$ it follows that a function $f = (f_{i,j})_{1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant N}$ belongs to the space $\mathscr{L}_n^\infty(\mu)$ if and only if each component $f_{i,j}$ belongs to the space $L^\infty(\tau\mu)$. However, for the spaces $\mathscr{L}_n^p(\mu)$, with $p \geqslant 1$, this property is not true. We next show an example of this.

EXAMPLE. Let us consider the matrix of measures

$$\mu=\frac{1}{2}\begin{pmatrix}1 & t-1\\ t-1 & 1\end{pmatrix}\chi_{[0,1]}(t)\,dm,$$

where dm denotes the Lebesgue measure in $[0,1]$. Since the determinant of $M(t)$ is equal to $\frac{1}{4}t(2-t)$, which is non-negative in $[0,1]$, the matrix of measures μ is positive definite. Its trace is $\chi_{[0,1]}(t)\,dm$, that is, the Lebesgue measure in $[0,1]$.

The singular value decomposition of the matrix $M(t)$ in $[0,1]$ is

$$M(t)=\frac{1}{2}\,P\begin{pmatrix}2-t & 0\\ 0 & t\end{pmatrix}P^*,\qquad \text{where}\quad P=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix},$$

and hence

$$M(t)^{2/p}=\frac{1}{2^{2/p}}\,P\begin{pmatrix}(2-t)^{2/p} & 0\\ 0 & t^{2/p}\end{pmatrix}P^*.$$

For the function

$$f(t)=\left(\frac{1}{t^{1/p}},\frac{1}{t^{1/p}}\right)$$

we have that

$$\begin{aligned}
&[f(t)\,M(t)^{2/p}\,f(t)^*]^{p/2}\\
&=\left[\frac{1}{2^{2/p}}\,\frac{1}{2}\begin{pmatrix}\frac{1}{t^{1/p}} & \frac{1}{t^{1/p}}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}(2-t)^{2/p} & 0\\ 0 & t^{2/p}\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{t^{1/p}}\\ \frac{1}{t^{1/p}}\end{pmatrix}\right]^{p/2}\\
&=\left[\frac{1}{2}\,\frac{1}{2^{p/2}}\begin{pmatrix}0 & \frac{2}{t^{1/p}}\end{pmatrix}\begin{pmatrix}(2-t)^{2/p} & 0\\ 0 & t^{2/p}\end{pmatrix}\begin{pmatrix}0\\ \frac{2}{t^{1/p}}\end{pmatrix}\right]^{p/2}\\
&=2^{p/2-1},
\end{aligned}$$

which is constant and thus integrable with respect to $\tau\mu=\chi_{[0,1]}(t)\,dm$. So $f\in\mathscr{L}_1^p(\mu)$, for $p\geqslant 1$. However, it is clear that none of the components of $f(t)$ belongs to the space $L^p(\tau\mu)$.

Observe that if $f=(f_{i,j})_{1\leqslant i\leqslant n,\,1\leqslant j\leqslant N}$, and we call f_i to the rows of f, we have for $p\geqslant 1$

$$\tau(f(t)\,M(t)^{2/p}\,f(t)^*)^{p/2}=\left(\sum_{i=1}^{n} f_i(t)\,M(t)^{2/p}\,f_i(t)^*\right)^{p/2}$$
$$=\left(\sum_{i=1}^{n}\|f_i(t)\,M(t)^{1/p}\|_E^2\right)^{p/2},$$

which is the norm $2/p$ in $\mathbb{R}^n$ of the vector

$$(\|f_1(t)\,M(t)^{1/p}\|_E^p, ..., \|f_n(t)\,M(t)^{1/p}\|_E^p),$$

where $\|\cdot\|_E$ denotes the 2 norm in $\mathbb{R}^N$.

By using now that norms $2/p$ and 1 are equivalent in $\mathbb{R}^n$, there exist two positive constants C_1 and C_2 such that

$$C_1\sum_{i=1}^{n}\|f_i(t)\,M(t)^{1/p}\|_E^p\leqslant\left(\sum_{i=1}^{n}\|f_i(t)\,M(t)^{1/p}\|_E^2\right)^{p/2}$$
$$\leqslant C_2\sum_{i=1}^{n}\|f_i(t)\,M(t)^{1/p}\|_E^p,$$

and from this formula we obtain by integrating that

$$C_1\sum_{i=1}^{n}\|f_i\|_{p,\mu}^p\leqslant\|f\|_{p,\mu}^p\leqslant C_2\sum_{i=1}^{n}\|f_i\|_{p,\mu}^p.$$

This property is interesting since it reduces the study of the spaces of matrix functions $\mathscr{L}_n^p(\mu)$ to the study of the space of vector functions $\mathscr{L}_1^p(\mu)$. For example, to study density questions in the spaces $\mathscr{L}_n^p(\mu)$, given a positive definite matrix of measures, the space of polynomials $\mathbb{C}^{n\times N}[x]$ is dense in $\mathscr{L}_n^p(\mu)$ if and only if the space of polynomials $\mathbb{C}^N[x]$ is dense in the space $\mathscr{L}_1^p(\mu)$, because if it is possible to approximate by polynomials in norm p all the rows of a function f, then it is possible to approximate by polynomials in norm p the function f, and vice versa. From now on, we denote by $\mathscr{L}^p(\mu)$ the space $\mathscr{L}_1^p(\mu)$.

We prove first that $\|\cdot\|_{p,\mu}$ is a norm in $\mathscr{L}_n^p(\mu)$. The following lemma is straightforwardly proved:

LEMMA 2.1. *If M is a positive semidefinite numerical matrix, then it defines an inner product in $M_{n\times N}(\mathbb{C})$ given by $(a,b)=\tau(aMb^*)$. Moreover, $\|a\|_M=(a,a)^{1/2}$ is a seminorm in $M_{n\times N}(\mathbb{C})$.*

From now on, if $a \in M_{n\times N}(\mathbb{C})$, M is a numerical positive semidefinite matrix, and $1 \leqslant p < \infty$, we use the following notation: $\|a\|_{M,p} = \tau(aM^{2/p}a^*)^{1/2}$. For $p=\infty$, we define $\|a\|_{M,\infty} = \tau(aPa^*)^{1/2}$, and P is as defined above. As a consequence of Lemma 2.1, $\|\cdot\|_{M,p}$ is a seminorm in $M_{n\times N}(\mathbb{C})$. The following lemma is also immediate:

LEMMA 2.2. *If M is a positive semidefinite matrix, $p \geqslant 1$ and $1/p + 1/q = 1$, then for any vectors a, b in $M_{n\times N}(\mathbb{C})$ we have*

$$|\tau(aMb^*)| \leqslant \|a\|_{M,p}\, \|b\|_{M,q}.$$

LEMMA 2.3 (Hölder's inequality). *If $1 \leqslant p < \infty$, $1/p + 1/q = 1$, $f \in \mathcal{L}^p_n(\mu)$, and $g \in \mathcal{L}^q_n(\mu)$, then $\tau(f(t)\, M(t)\, g(t)^*) \in L^1(\tau\mu)$, and*

$$\|\tau(fMg^*)\|_{1,\tau\mu} \leqslant \|f\|_{p,\mu}\, \|g\|_{q,\mu}.$$

Proof. It is enough to proceed as in the scalar case, using Lemma 2.2 ∎

We next prove that $\mathcal{L}^p_n(\mu)$ is a Banach space.

THEOREM 2.4. *If $1 \leqslant p \leqslant \infty$, the space $\mathcal{L}^p_n(\mu)$ is a Banach space, Hilbert for $p = 2$.*

Proof. To prove that $\|\cdot\|_{p,\mu}$ is a norm it is again enough to proceed as in the scalar case.

Given $\mu = (\mu_{i,j})_{i,j=1}^N$ a positive definite matrix of measures and $M = (m_{i,j})_{i,j=1}^N$ its Radon–Nikodym derivative, the eigenvalues $\lambda_i(t)$ associated to $M(t)$ are real and non-negative because $M(t)$ is positive semidefinite for any real number t. Moreover, from $\theta \leqslant M(t) \leqslant I$ we deduce that $0 \leqslant \lambda_i(t) \leqslant 1$, for any eigenvalue $\lambda_i(t)$ of $M(t)$. We call $0 \leqslant \lambda_1(t) \leqslant \cdots \leqslant \lambda_N(t) \leqslant 1$ to the eigenvalues of $M(t)$ and $v_1(t), ..., v_N(t)$ to a basis of $\mathbb{C}^N$ formed with eigenvectors of $M(t)$ associated to $\lambda_1(t), ..., \lambda_N(t)$, respectively. We can choose the eigenvectors satisfying $v_i(t)\, v_j(t)^* = \delta_{i,j}$.

Given a function $f\colon \mathbb{R} \to M_{n\times N}(\mathbb{C})$, we can express it in the following way,

$$f(t) = \begin{pmatrix} \sum_{i=1}^N \alpha_{1,i}(t)\, v_i(t) \\ \vdots \\ \sum_{i=1}^N \alpha_{n,i}(t)\, v_i(t) \end{pmatrix}, \tag{2.1}$$

where $\alpha_{j,i}(t)$ are measurable functions, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant n$. Observe that if for $M(t_0)$ the first k eigenvectors $v_1(t_0), ..., v_k(t_0)$ are associated to the eigenvalue 0, the values $\alpha_{j,1}(t_0), ..., \alpha_{j,k}(t_0)$ are not important, since for the spaces $\mathscr{L}_n^p(\mu)$, $1 \leqslant p \leqslant \infty$, $v_i(t_0)$ can be identified with the null vector, $1 \leqslant i \leqslant k$. We assume that these values are 0.

Suppose first that $1 \leqslant p < \infty$. According to the definition, $f \in \mathscr{L}_n^p(\mu)$ if and only if $\tau(f(t)\, M(t)^{2/p}\, f(t)^*)^{1/2} \in L^p(\tau\mu)$.

We have

$$\tau(f(t)\, M(t)^{2/p}\, f(t)^*)^{p/2} = \left(\sum_{j=1}^{n} \sum_{i=1}^{N} \lambda_i(t)^{2/p}\, |\alpha_{j,i}(t)|^2\, |v_i(t)|_E^2 \right)^{p/2}$$

$$= \left(\sum_{j=1}^{n} \sum_{i=1}^{N} \lambda_i(t)^{2/p}\, |\alpha_{j,i}(t)|^2 \right)^{p/2}, \tag{2.2}$$

which is the norm $2/p$ of the $n \times N$ matrix

$$\begin{pmatrix} \lambda_1(t)\, |\alpha_{1,1}(t)|^p & \cdots & \lambda_N(t)\, |\alpha_{1,N}(t)|^p \\ \vdots & \ddots & \vdots \\ \lambda_1(t)\, |\alpha_{n,1}(t)|^p & \cdots & \lambda_N(t)\, |\alpha_{n,N}(t)|^p \end{pmatrix}.$$

Since the norms $2/p$ and 1 are equivalent in $\mathbb{R}^{n \times N}$, there exist two positive constants C_1 and C_2 such that

$$C_1 \sum_{j=1}^{n} \sum_{i=1}^{N} \lambda_i(t)\, |\alpha_{j,i}(t)|^p \leqslant \left(\sum_{j=1}^{n} \sum_{i=1}^{N} \lambda_i(t)^{2/p}\, |\alpha_{j,i}(t)|^2 \right)^{p/2}$$

$$\leqslant C_2 \sum_{j=1}^{n} \sum_{i=1}^{N} \lambda_i(t)\, |\alpha_{j,i}(t)|^p$$

and hence, $f \in \mathscr{L}_n^p(\mu)$ is equivalent to $\lambda_i(t)^{1/p}\, \alpha_{j,i}(t) \in L^p(\tau\mu)$, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant n$. If we consider the space

$$\Theta_p = \bigoplus_{i=1}^{N} L^p(\tau\mu\chi_{[\lambda_i \neq 0]}, \mathbb{C}^n)$$

whose elements are matrix functions $(h_{j,i})_{1 \leqslant j \leqslant n,\, 1 \leqslant i \leqslant N}$ such that $h_{j,i} \in L^p(\tau\mu\chi_{[\lambda_i \neq 0]})$, $\forall i, j$, we can define an operator $T_p \colon \mathscr{L}_n^p(\mu) \to \Theta_p$ in the following way: if we express f as in (2.1), we define

$$T_p(f) = \begin{pmatrix} \lambda_1(t)^{1/p}\, \alpha_{1,1}(t) & \cdots & \lambda_N(t)^{1/p}\, \alpha_{1,N}(t) \\ \vdots & \ddots & \vdots \\ \lambda_1(t)^{1/p}\, \alpha_{n,1}(t) & \cdots & \lambda_N(t)^{1/p}\, \alpha_{n,N}(t) \end{pmatrix}.$$

It is clear that T_p is linear and bijective between $\mathcal{L}_n^p(\mu)$ and Θ_p, and if we consider in the space Θ_p the norm $\|\cdot\|_{\Theta_p}$ given by

$$\|h\|_{\Theta_p} = \| \, \|h(t)\|_E \|_{p,\, \tau\mu} = \left(\int_{\mathbb{R}} \tau(h(t)\, h(t)^*)^{p/2} \, d\tau\mu \right)^{1/p},$$

we have that T_p is an isometry:

$$\begin{aligned}
\|T_p f\|_{\Theta_p} &= \left(\int_{\mathbb{R}} \|T_p f(t)\|_E^p \, d\tau\mu(t) \right)^{1/p} \\
&= \left(\int_{\mathbb{R}} \left(\sum_{j=1}^{n} \sum_{i=1}^{N} \lambda_i(t)^{2/p} \, |\alpha_{j,\,i}(t)|^2 \right)^{p/2} d\tau\mu(t) \right)^{1/p} \\
&= \left(\int_{\mathbb{R}} \tau(f(t)\, M(t)^{2/p}\, f(t)^*)^{p/2} \, d\tau\mu(t) \right)^{1/p} \\
&= \|f\|_{p,\,\mu}.
\end{aligned}$$

Hence, we have that $\mathcal{L}_n^p(\mu)$ is isometric to Θ_p.

For $p=\infty$, from expression (2.1) we deduce that $f(t) \in \mathcal{L}_n^\infty(\mu)$ if and only if $|\alpha_{j,\,i}(t)|^2 \, \chi_{[\lambda_i \neq 0]}(t) \in L^\infty(\tau\mu)$, for $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant n$. Now we can consider the space

$$\Theta_\infty = \bigoplus_{i=1}^{n} L^\infty(\tau\mu\chi_{[\lambda_i \neq 0]}, \mathbb{C}^n)$$

and prove similarly that $\mathcal{L}_n^\infty(\mu)$ is isometric to Θ_∞.

Since the spaces Θ_p are Banach spaces for $1 \leqslant p \leqslant \infty$ (Hilbert if $p=2$), we deduce that the spaces $\mathcal{L}_n^p(\mu)$ are Banach spaces for $1 \leqslant p \leqslant \infty$ (Hilbert if $p=2$). ∎

We now show how this identification allows the calculation of the dual of the spaces $\mathcal{L}_n^p(\mu)$ in a simple way.

THEOREM 2.5. *If $1 \leqslant p < \infty$ and q is its conjugate, that is, $1/p + 1/q = 1$, $g \in \mathcal{L}_n^q(\mu)$ and we define $\Lambda_g \colon \mathcal{L}_n^p(\mu) \to \mathbb{C}$ by*

$$\Lambda_g(f) = \int_{\mathbb{R}} \tau(fMg^*) \, d\tau\mu, \qquad \text{for every} \quad f \in \mathcal{L}_n^p(\mu), \tag{2.3}$$

then Λ is a linear and continuous mapping, and $\|\Lambda\| \leqslant \|g\|_{q,\,\mu}$. Furthermore, if $\Lambda \colon \mathcal{L}_n^p(\mu) \to \mathbb{C}$ is linear and continuous, then there exists a unique g in $\mathcal{L}_n^q(\mu)$ such that Λ can be expressed as in (2.3), and $\|\Lambda\| = \|g\|_{q,\,\mu}$.

If $\tau\mu$ is σ-finite, then this result holds also for $\mathcal{L}_n^\infty(\mu)$.

Proof. It is clear that Λ is linear, and using Hölder's inequality

$$|\Lambda(f)| \leqslant \int_{\mathbb{R}} |fMg^*| \, d\tau\mu \leqslant \|f\|_{p,\mu} \, \|g\|_{q,\mu}.$$

Moreover,

$$\|\Lambda\| = \sup_{\|f\|_{p,\mu} \leqslant 1} |\Lambda(f)| \leqslant \|g\|_{q,\mu}.$$

We prove now that for $1 \leqslant p < \infty$, any linear and continuous mapping from $\mathscr{L}_n^p(\mu)$ into $\mathbb{C}$ can be represented as in (2.3), for a certain unique function $g \in \mathscr{L}_n^q(\mu)$.

Suppose first that $1 < p < \infty$ and $\Lambda: \mathscr{L}_n^p(\mu) \to \mathbb{C}$ is a linear and continuous mapping. T_p is an isometry; hence the composition $\Lambda T_p^{-1}: \Theta_p \to \mathbb{C}$ is also a linear and continuous mapping. The dual space of Θ_p is isometric to Θ_q; hence there exists a function $G = (G_{j,i})_{1 \leqslant j \leqslant n,\, 1 \leqslant i \leqslant N} \in \Theta_q$ such that for any function $F = (F_{j,i})_{1 \leqslant j \leqslant n,\, 1 \leqslant i \leqslant N} \in \Theta_p$ we have

$$\Lambda T_p^{-1}(F) = \int_{\mathbb{R}} \left(\sum_{j=1}^{n} \sum_{i=1}^{N} F_{j,i} \overline{G_{j,i}} \right) d\tau\mu,$$

and $\|G\|_{\Theta_q} = \|\Lambda T_p^{-1}\|$. Hence, for any function $f \in \mathscr{L}_n^p(\mu)$, $f = (f_{j,i})_{1 \leqslant j \leqslant n,\, 1 \leqslant i \leqslant N}$ we have

$$\Lambda(f) = \Lambda T_p^{-1}(T_p f) = \int_{\mathbb{R}} \left(\sum_{j=1}^{n} \sum_{i=1}^{N} \lambda_i^{1/p} f_{j,i} \overline{G_{j,i}} \right) d\tau\mu. \tag{2.4}$$

We put $g = T_q^{-1}(G) \in \mathscr{L}_n^q(\mu)$, that is, $G = T_q(g)$. Equation (2.4) can now be written as

$$\begin{aligned} \Lambda(f) &= \int_{\mathbb{R}} \left(\sum_{j=1}^{n} \sum_{i=1}^{N} \lambda_i^{1/p} f_{j,i} \overline{g_{j,i}} \lambda_i^{1/q} \right) d\tau\mu \\ &= \int_{\mathbb{R}} \left(\sum_{j=1}^{n} \sum_{i=1}^{N} \lambda_i f_i \overline{g_i} \right) d\tau\mu \\ &= \int_{\mathbb{R}} \tau(fMg^*) \, d\tau\mu, \end{aligned}$$

and we also have

$$\begin{aligned} \|\Lambda\| &= \sup_{\|f\|_{p,\mu} = 1} |\Lambda(f)| = \sup_{\|F\|_{\Theta_p} = 1} |\Lambda T_p^{-1}(F)| = \|\Lambda T_p^{-1}\| \\ &= \|G\|_{\Theta_q} = \|T_q(g)\|_{\Theta_q} = \|g\|_{q,\mu}. \end{aligned}$$

Hence the dual space of $\mathscr{L}_n^p(\mu)$ is isometric to $\mathscr{L}_n^q(\mu)$.

For $p=\infty$, it is enough to proceed in the same way. ∎

To complete the study of the spaces $\mathscr{L}_n^p(\mu)$, we prove the continuity of the inclusions $\mathscr{L}_n^p(\mu)\subseteq\mathscr{L}_n^q(\mu)$ $(q>p)$ when $\tau\mu(\mathbb{R})<\infty$.

THEOREM 2.6. *If μ is a positive definite matrix of measures such that $\tau\mu(\mathbb{R})<\infty$, then*

(1) *$\mathscr{L}_n^\infty(\mu)\subseteq\mathscr{L}_n^1(\mu)$ and the inclusion is continuous, since for any function f in $\mathscr{L}_n^\infty(\mu)$ we have*

$$\|f\|_{1,\mu}\leqslant\tau\mu(\mathbb{R})\,\|f\|_{\infty,\mu}.$$

(2) *If $1\leqslant p<q$, $\mathscr{L}_n^p(\mu)\subseteq\mathscr{L}_n^q(\mu)$ and also in this case the inclusion is continuous, since for any function f in $\mathscr{L}_n^p(\mu)$ we have*

$$\|f\|_{q,\mu}\leqslant\tau\mu(\mathbb{R})^{1/q-1/p}\,\|f\|_{p,\mu}.$$

Proof. (1) Taking into account that $\theta\leqslant M^2\leqslant M\leqslant P\leqslant I$ we have that given a function f in $\mathscr{L}_n^\infty(\mu)$,

$$\|f\|_{1,\mu}=\int_{\mathbb{R}}\tau(fM^2f^*)^{1/2}\,d\tau\mu\leqslant\int_{\mathbb{R}}\tau(fPf^*)^{1/2}\,d\tau\mu\leqslant\tau\mu(\mathbb{R})\,\|f\|_{\infty,\mu}.$$

(2) From $p>q$ and $\theta\leqslant M\leqslant I$ we deduce $\theta\leqslant M^{1/q}\leqslant M^{1/p}$. Given f a function in $\mathscr{L}_n^p(\mu)$, we have that the function $\tau(fM^{2/p}f^*)^{q/2}\in L^{p/q}(\tau\mu)$, and since $\tau\mu(\mathbb{R})<\infty$, the constant function $1\in L^{p/(q-p)}(\tau\mu)$ $(p/(p-q)$ is the conjugate exponent of $p/q)$. From the inequality for M and using Hölder's inequality in the scalar case we deduce

$$\begin{aligned}\|f\|_{q,\mu}^q&=\int_{\mathbb{R}}\tau(fM^{2/q}f^*)^{q/2}\,d\tau\mu\leqslant\int_{\mathbb{R}}\tau(fM^{2/p}f^*)^{q/2}\,d\tau\mu\\&\leqslant\left(\int_{\mathbb{R}}d\tau\mu\right)^{(p-q)/p}\left(\int_{\mathbb{R}}\tau(fM^{2/p}f^*)^{p/2}\,d\tau\mu\right)^{q/p}\\&=\tau\mu(\mathbb{R})^{(p-q)/p}\,\|f\|_{p,\mu}^q.\end{aligned}$$

By taking now qth root we obtain the result. ∎

3. COMPACTNESS OF THE SET OF SOLUTIONS OF A MATRIX MOMENT PROBLEM

In this section we study some topological properties of the space of matrices of measures. As a consequence it will be proved that the set of solutions for a matrix moment problem is, as in the scalar case, a compact convex set. This result will guarantee the existence of extremal points in the set of solutions, which will be of some utility in studying the density of polynomials in the space $\mathscr{L}^1$ of a matrix of measures.

We put $\mathscr{M}_N$ for the set of positive definite matrices of measures, and $\mathscr{M}_N^*$ for the set of positive definite matrices of measures with finite moments of any order.

Given $\mu \in \mathscr{M}_N^*$, we consider the set $[\mu]$ defined by

$$[\mu] = \left\{ \nu \in \mathscr{M}_N^* \text{ such that } \int_{\mathbb{R}} t^n \, d\nu_{i,j} = \int_{\mathbb{R}} t^n \, d\mu_{i,j}, \right.$$
$$\left. \text{for any } n \in \mathbb{N} \text{ and for } 1 \leqslant i, j \leqslant N \right\},$$

that is, the set of positive definite matrices of measures whose moments are the same as those of μ, or equivalently, the set of solutions for the matrix moment problem defined by μ.

The vague topology on $\mathscr{M}_N$ is the coarsest topology for which the mappings $\mu \to \int_{\mathbb{R}} f \, d\mu$ are continuous, where $f \in \mathscr{C}_c(\mathbb{R})$ is arbitrary. $\mathscr{C}_c(\mathbb{R})$ denotes the set of continuous functions with compact support defined on $\mathbb{R}$. The weak topology on $\mathscr{M}_N$ is the coarsest topology for which the mappings $\mu \to \int_{\mathbb{R}} f d\mu$ are continuous, where $f \in \mathscr{C}_b(\mathbb{R})$ is arbitrary. $\mathscr{C}_b(\mathbb{R})$ denotes the set of continuous and bounded functions defined on $\mathbb{R}$.

Since $\mathscr{C}(\mathbb{R})$ is strictly included in $\mathscr{C}_c(\mathbb{R})$ it is clear that the vague topology is finer, that is, it has less open sets, than the weak topology. It is not hard to see that both topologies are Hausdorff.

If $A \subseteq C_c(\mathbb{R})$ is an arbitrary set spanning $\mathscr{C}_c(\mathbb{R})$ then the vague topology is also the coarsest for which all the mappings $\mu \to \int_{\mathbb{R}} f \, d\mu$ are continuous, where f ranges through A. We will use this remark for $A = \mathscr{C}_c^+(\mathbb{R})$.

In our first result we establish the relationship between the vague and weak convergence.

THEOREM 3.1. *Given $(\mu_\alpha)_{\alpha \in A}$ and μ in $\mathscr{M}_N$ finite (that is, $\tau\mu_\alpha(\mathbb{R}) < \infty$ and $\tau\mu(\mathbb{R}) < \infty$) we have that $\mu_\alpha \to \mu$ weakly if and only if $\mu_\alpha \to \mu$ vaguely and $\lim_\alpha \tau\mu_\alpha(\mathbb{R}) = \tau\mu(\mathbb{R})$.*

The proof of Theorem 3.1 follows from the following lemmas.

LEMMA 3.2. *Suppose $(\mu_\alpha)_{\alpha \in A}$, μ in $\mathscr{M}_N$, $\mu_\alpha \to \mu$ vaguely, and $g: \mathbb{R} \to [0, \infty)$ is a continuous function. Then*

(1) *If $\limsup_{\alpha \in A} \int_{\mathbb{R}} g \, d\tau\mu_\alpha < \infty$ then*

$$\int_{\mathbb{R}} g \, d\tau\mu < \infty \qquad \text{and} \qquad \lim_\alpha \int_{\mathbb{R}} f \, d\mu_{\alpha,i,j} = \int_{\mathbb{R}} f \, d\mu_{i,j},$$

for any $f \in o(g)$ and for any $1 \leqslant i, j \leqslant N$.

(2) *If* $\lim_{\alpha\in A}\int_{\mathbb{R}} g\, d\tau\mu_\alpha=\int_{\mathbb{R}} g\, d\tau\mu<\infty$ *then*

$$\lim_\alpha \int_{\mathbb{R}} f\, d\mu_{\alpha, i, j}=\int_{\mathbb{R}} f\, d\mu_{i, j},$$

for any $f\in O(g)$ *and for any* $1\leqslant i, j\leqslant N$.

Proof. The first part of (1) follows as in the scalar case.

The second part of (1) is also analogous to the scalar case, and follows by using that for a positive definite matrix of measures the trace bounds all the components. Suppose now that $f\in o(g)$ and $\varepsilon>0$ are given. There exists a compact set K such that $|f(x)|\leqslant \varepsilon g(x)$, for any $x\in K^C$. Let $\phi\in C_c^+(\mathbb{R})$ be chosen such that $\chi_K\leqslant\phi\leqslant 1$. Then we have $|f|\,(1-\phi)\leqslant\varepsilon g$ and hence

$$\begin{aligned}
&\left|\int_{\mathbb{R}} f\, d\mu_{i, j}-\int_{\mathbb{R}} f\, d\mu_{\alpha, i, j}\right| \\
&\quad\leqslant \left|\int_{\mathbb{R}} f\, d\mu_{i, j}-\int_{\mathbb{R}} f\phi\, d\mu_{i, j}\right|+\left|\int_{\mathbb{R}} f\phi\, d\mu_{i, j}-\int_{\mathbb{R}} f\phi\, d\mu_{\alpha, i, j}\right| \\
&\quad\quad+\left|\int_{\mathbb{R}} f\phi\, d\mu_{\alpha, i, j}-\int_{\mathbb{R}} f\, d\mu_{\alpha, i, j}\right| \\
&\quad\leqslant \varepsilon\int_{\mathbb{R}} g\, d\,|\mu_{i, j}|+\left|\int_{\mathbb{R}} f\phi\, d(\mu_{i, j}-\mu_{\alpha, i, j})\right|+\varepsilon\int_{\mathbb{R}} g\, d\,|\mu_{\alpha, i, j}| \\
&\quad\leqslant \varepsilon\int_{\mathbb{R}} g\, d\tau\mu+\left|\int_{\mathbb{R}} f\phi\, d(\mu_{i, j}-\mu_{\alpha, i, j})\right|+\varepsilon\int_{\mathbb{R}} g\, d\tau\mu_\alpha.
\end{aligned}$$

Taking $\limsup_\alpha$ yields

$$\limsup_\alpha \left|\int_{\mathbb{R}} f\, d\mu_{i, j}-\int_{\mathbb{R}} f\, d\mu_{\alpha, i, j}\right|\leqslant\varepsilon(c_1+c_2)$$

and since ε was arbitrary we deduce that

$$\limsup_\alpha \left|\int_{\mathbb{R}} f\, d\mu_{i, j}-\int_{\mathbb{R}} f\, d\mu_{\alpha, i, j}\right|=0$$

which gives the result.

(2) The proof of (2) is also analogous to the scalar case, using the same inequalities as above and that the trace bounds the components of a positive definite matrix of measures. ∎

The following lemma is a consequence of the previous one:

LEMMA 3.3. *Suppose $\mu_\alpha \to \mu$ vaguely. We have:*

(1) *If* $\limsup_\alpha \tau\mu_\alpha(\mathbb{R}) < \infty$ *then* $\tau\mu(\mathbb{R}) < \infty$ *and*

$$\lim_\alpha \int_{\mathbb{R}} f\, d\mu_{\alpha, i, j} = \int_{\mathbb{R}} f\, d\mu_{i, j},$$

for any $f \in o(1)$ and for any $1 \leqslant i, j \leqslant N$.

(2) *If* $\lim_\alpha \tau\mu_\alpha(\mathbb{R}) = \tau\mu(\mathbb{R})$ *then*

$$\lim_\alpha \int_{\mathbb{R}} f\, d\mu_{\alpha, i, j} = \int_{\mathbb{R}} f\, d\mu_{i, j},$$

for any $f \in O(1)$ and for any $1 \leqslant i, j \leqslant N$.

Proof of Theorem 3.1. It is clear that if $\mu_\alpha \to \mu$ weakly then $\mu_\alpha \to \mu$ vaguely, and furthermore, since $\tau\mu_\alpha \to \tau\mu$ weakly, taking $f = 1$ in the definition of weak convergence we have $\tau\mu_\alpha(\mathbb{R}) \to \tau\mu(\mathbb{R})$.

On the other hand, if $\mu_\alpha \to \mu$ vaguely and $\tau\mu_\alpha(\mathbb{R}) \to \tau\mu(\mathbb{R})$, from part (2) of Lemma 3.3 and since all the functions in $\mathscr{C}_b(\mathbb{R})$ are bounded and thus belong to $O(1)$, we deduce that $\mu_\alpha \to \mu$ weakly. ∎

It is of some interest to know if the vague topology on $\mathscr{M}_N$ is metrizable, because then the topological notions can be expressed in terms of sequences instead of nets. We have the following theorem:

THEOREM 3.4. *$\mathscr{M}_N$ is metrizable in the vague topology.*

The proof is not very different from the one given in [1] for the one-dimensional case. In Problems 1.1 of Section 7.7 and 2 of Section 7.8 of [1] it is proved that given a sequence of functions $\mathscr{D} = (\eta_n)_n$ in $\mathscr{C}_c(\mathbb{R})$ satisfying:

(1) For any compact set $K \subseteq \mathbb{R}$ there exists a relatively compact neighbourhood U of K such that every function $f \in \mathscr{C}_c(\mathbb{R})$ with $\operatorname{supp} f \subseteq K$ is uniformly approximable in $\mathbb{R}$ by functions in $\mathscr{D}$ whose support lies inside U.

(2) For those K and U, there exists a η_n with $0 \leqslant \chi_K \leqslant \eta_n \leqslant 1$.

we have that a sequence of measures μ_n converges vaguely to μ if and only if

$$\lim_{n \to \infty} \int_{\mathbb{R}} \eta\, d\mu_n = \int_{\mathbb{R}} \eta\, d\mu, \qquad \text{for every} \quad \eta \in \mathscr{D}.$$

Furthermore, if we define

$$\rho(\mu, \nu) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min \left\{ 1, \left| \int_{\mathbb{R}} \eta_n \, d\mu - \int_{\mathbb{R}} \eta_n \, d\nu \right| \right\},$$

ρ is a metric on $\mathscr{M}_1$ and the corresponding topology is the vague topology. In a similar way it is possible to prove that $\mathscr{M}_N$ is metrizable, defining for two matrices of measures $\mu = (\mu_{i,j})_{1 \leqslant i,j \leqslant N}$, $\nu = (\nu_{i,j})_{1 \leqslant i,j \leqslant N}$

$$\rho(\mu, \nu) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min \left\{ 1, \max_{1 \leqslant i,j \leqslant N} \left| \int_{\mathbb{R}} \eta_n \, d\mu_{i,j} - \int_{\mathbb{R}} \eta_n \, d\nu_{i,j} \right| \right\}.$$

ρ is a metric on $\mathscr{M}_N$ and the corresponding topology is the vague topology.

We prove next that the positive definite character of matrices of measures is preserved under weak and vague limit:

THEOREM 3.5. *If $(\mu_n)_n$ are finite matrices of measures in $\mathscr{M}_N$ and $\mu = (\mu_{i,j})_{1 \leqslant i,j \leqslant N}$ is a finite matrix of measures, and $\mu_n \to \mu$ vaguely, then μ is positive definite. That is, the positive definite character is preserved under vague limit.*

To prove this theorem we will use the following lemma:

LEMMA 3.6. *Given $\mu = (\mu_{i,j})_{1 \leqslant i,j \leqslant N}$ a finite matrix of measures, the following properties are equivalent*:

(1) *For any Borel set A the numerical matrix $(\mu_{i,j}(A))_{1 \leqslant i,j \leqslant N}$ is positive semidefinite.*

(2) *For any $f \in C_c^+(\mathbb{R})$, the numerical matrix $(\int_{\mathbb{R}} f \, d\mu_{i,j})_{1 \leqslant i,j \leqslant N}$ is positive semidefinite.*

Proof. (2) $\Rightarrow$ (1) Given an interval $(a, b) \subseteq \mathbb{R}$, we consider f_n an increasing sequence of functions in $\mathscr{C}_c^+(\mathbb{R})$ such that $f_n \to \chi_{(a,b)}$ pointwise. By considering the Hahn decomposition for the matrix of measures μ: $\mu = \mu^1 - \mu^2 + i(\mu^3 - \mu^4)$, where all the measures in the matrix of measures μ^i are positive measures and using that these measures are bounded by the measure $\tau\mu$, the monotone convergence theorem gives that for any $1 \leqslant i, j \leqslant N$,

$$\mu((a, b)) = \int_{\mathbb{R}} \chi_{(a,b)}(t) \, d\mu(t) = \lim_{n \to \infty} \int_{\mathbb{R}} f_n(t) \, d\mu(t).$$

Since all the numerical matrices $\int_{\mathbb{R}} f_n(t) \, d\mu(t)$ are positive semidefinite $\mu((a, b))$ is also a positive semidefinite matrix. By using now the regularity of μ we deduce that $\mu(A)$ is positive semidefinite, for any Borel set A.

Suppose now (1) and let us prove (2). Suppose on the contrary that there exists $f \in C_c^+(\mathbb{R})$ such that the numerical matrix $(\int_{\mathbb{R}} f \, d\mu_{i,j})_{1 \leqslant i, j \leqslant N}$ is not positive semidefinite; that is, there exists a non-zero vector $c = (c_1, ..., c_N)$ such that $c(\int_{\mathbb{R}} f \, d\mu_{i,j}) \, c^* \ngeqslant 0$. Let us put

$$d = \operatorname{dist} \left\{ c \left(\int_{\mathbb{R}} f \, d\mu_{i,j} \right) c^*, \{ x \in \mathbb{R}, x \geqslant 0 \} \right\} > 0.$$

Clearly,

$$d \geqslant \left| \Im c \left(\int_{\mathbb{R}} f \, d\mu_{i,j} \right) c^* \right|. \tag{3.1}$$

Since $f \in C_c^+(\mathbb{R})$, we can find a simple function $f_0 = \sum_{k=1}^{k_0} a_i \chi_{A_k}$, with $a_k \geqslant 0$, which approximates f in the following way:

$$\| f - f_0 \|_\infty \leqslant \frac{d}{2N^2 \max_{1 \leqslant i \leqslant N} |c_i|^2 \max_{1 \leqslant i, j \leqslant N} \|\mu_{i,j}\|}.$$

From the equality

$$c \left(\int_{\mathbb{R}} f \, d\mu_{i,j} \right) c^* = c \left(\int_{\mathbb{R}} (f - f_0) \, d\mu_{i,j} \right) c^* + c \left(\int_{\mathbb{R}} f_0 \, d\mu_{i,j} \right) c^*,$$

taking into account that

$$\left| c \left(\int_{\mathbb{R}} (f - f_0) \, d\mu_{i,j} \right) c^* \right| \leqslant 2N^2 \max_{1 \leqslant i \leqslant N} |c_i|^2 \max_{1 \leqslant i, j \leqslant N} \|\mu_{i,j}\| \, \|f - f_0\|_\infty \leqslant \frac{d}{2}$$

and that

$$c \left(\int_{\mathbb{R}} f_0 \, d\mu_{i,j} \right) c^* = c \left(\sum_{k=1}^{k_0} a_k \mu_{i,j}(A_k) \right) c^* = \sum_{k=1}^{k_0} a_k c \mu_{i,j}(A_k) \, c^* \geqslant 0$$

we deduce that

$$\left| \Im c \left(\int_{\mathbb{R}} f \, d\mu_{i,j} \right) c^* \right| = \left| \Im c \left(\int_{\mathbb{R}} (f - f_0) \, d\mu_{i,j} \right) c^* \right| \leqslant \left| c \left(\int_{\mathbb{R}} (f - f_0) \, d\mu_{i,j} \right) c^* \right| \leqslant \frac{d}{2}$$

which contradicts (3.1). ∎

Proof of Theorem 3.5. Suppose $\mu_n \to \mu$ vaguely, where $\mu_n \in \mathscr{M}_N$. We must prove that the matrix of measures μ is positive definite. Suppose on the contrary that μ is not positive definite; then by Lemma 3.6 there exists $f \in C_c^+(\mathbb{R})$ such that the numerical matrix $(\int_{\mathbb{R}} f\, d\mu)$ is not positive semi-definite. Now the proof finishes like the proof of (1) $\Rightarrow$ (2) in Lemma 3.6. ∎

To finish this section we prove that the set of solutions to a matrix moment problem is vaguely and weakly compact and convex.

THEOREM 3.7. *$[\mu]$ is a compact convex set in the weak and vague topology coinciding on $[\mu]$.*

This result is an immediate consequence of the three following lemmas.

LEMMA 3.8. *The set $\{\mu \in \mathcal{M}_N : \tau\mu(\mathbb{R}) \leqslant 1\}$ is vaguely compact.*

Proof. Given $(\mu_n)_{n \in \mathbb{N}}$ in $\{\mu \in \mathcal{M}_N : \tau\mu(\mathbb{R}) \leqslant 1\}$, we decompose again $\mu_n = \mu_n^1 - \mu_n^2 + i(\mu_n^3 - \mu_n^4)$, μ_n^i being matrices of measures in which all the entries are positive measures. Since $|\mu_{n,i,j}| \leqslant \tau\mu_n$ for every n we deduce that all these measures are bounded by a fixed constant, and since the set of positive measures $\{\mu \geqslant 0, \|\mu\| \leqslant a\}$ is vaguely compact (see, for instance, [1]) we can find $\mu_0 = (\mu_{0,i,j})_{1 \leqslant i,j \leqslant N}$ such that $\mu_n \to \mu_0$ vaguely. It follows by Theorem 3.5 that μ_0 is a positive definite matrix of measures. $\tau\mu_0(\mathbb{R}) \leqslant 1$ is immediate by using that $\tau\mu_n \to \tau\mu_0$ vaguely. ∎

LEMMA 3.9. *Given μ a matrix of measures in $\mathcal{M}_N^*$, $[\mu]$ is a relatively vaguely compact set.*

Proof. For every $\nu \in [\mu]$ it is clear that $\tau\nu(\mathbb{R}) = \tau\mu(\mathbb{R})$. By Lemma 3.8, the set $\{\nu \in \mathcal{M}_N : \tau\nu(\mathbb{R}) \leqslant \tau\mu(\mathbb{R})\}$ is vaguely compact, and hence vaguely closed. We have $[\mu] \subseteq \{\nu \in \mathcal{M}_N : \tau\nu(\mathbb{R}) \leqslant \tau\mu(\mathbb{R})\}$, so by taking closure we get $\overline{[\mu]} \subseteq \{\nu \in \mathcal{M}_N : \tau\nu(\mathbb{R}) \leqslant \tau\mu(\mathbb{R})\}$. Now the set $\overline{[\mu]}$ is a closed set inside a compact set and consequently it is compact. ∎

LEMMA 3.10. *Given μ a matrix of measures in $\mathcal{M}_N^*$, the set $[\mu]$ is a vaguely closed set.*

Proof. If $\mu_n \to \mu$ vaguely, then by Theorem 3.5 $\mu \geqslant 0$, and taking into account that all the matrices of measures μ_n have the same moments, we deduce that $\limsup_n \int_{\mathbb{R}} t^{2n_0}\, d\tau\mu_n < \infty$ and hence for any polynomial $p(t) \geqslant 0$ we have $\limsup_n \int_{\mathbb{R}} p(t)\, d\tau\mu_n < \infty$.

Given a natural number n_0 we consider the polynomial $p(t) = (1 + t^{2n_0})(1 + t^2)$. Clearly $t^{n_0} \in o(p(t))$ and hence Lemma 3.2 gives

$$\lim_n \int_{\mathbb{R}} t^{n_0}\, d\mu_{n,i,j} = \int_{\mathbb{R}} t^{n_0}\, d\mu_{i,j} \qquad \text{for every} \quad 1 \leqslant i, j \leqslant N;$$

that is, the moments of μ coincide with those of μ_n. ∎

Proof of Theorem 3.7. The vague and weak topologies coincide on $[\mu]$ because if μ_α and μ belong to $[\mu]$, and $\mu_\alpha \to \mu$ vaguely, since the moments of $\tau\mu_\alpha$ and $\tau\mu$ are the same, the condition $\tau\mu(\mathbb{R})_\alpha \to \tau\mu(\mathbb{R})$ is trivially fulfilled, and according to Theorem 3.1 this implies that $\mu_\alpha \to \mu$. From Lemmas 3.9 and 3.10 it follows that $[\mu]$ is compact. Finally, it is clear that $[\mu]$ is convex. ∎

4. A DENSITY THEOREM FOR $\mathscr{L}^1(\mu)$

We now extend Naimark's theorem about density of polynomials in $\mathscr{L}^1_n(\mu)$. Naimark's theorem states that the polynomials are dense in $L^1(\mu)$, μ is a positive measure with moments of any order, if and only if μ is an extremal point of the convex set $\{v \geqslant 0 \colon \int t^n\, dv = \int t^n\, d\mu,\ n \geqslant 0\}$. So let μ be a positive definite matrix of measures with moments of any order so that the polynomials are included in $\mathscr{L}^1_n(\mu)$.

In the scalar case the trace of μ coincides with μ and then the matrix of Radon–Nikodym derivatives is $M(t) = 1$. Observe that the eigenvalues of $M(t)$ are in this case reduced to 1. In the matrix case the eigenvalues of $M(t)$ have a more complicated structure. To extend Naimark's theorem to the matrix case we need as an additional hypothesis that the eigenvalues of $M(t)$ do not approach 0, as in the scalar case:

Proof of Theorem 1. It is enough to prove the theorem for $n = 1$.

To prove that (1) implies (2), we suppose that the polynomials are not dense in $\mathscr{L}^1(\mu)$ and we prove that μ is not extremal in $[\mu]$. In that case, we can find a non-zero functional Λ in the dual space of $\mathscr{L}^1(\mu)$, with norm 1, such that $\Lambda(p) = 0$ for any polynomial p. From the duality theorem for $\mathscr{L}^1(\mu)$ we can represent this functional with a function $g = (g_1, ..., g_N)$ in $\mathscr{L}^\infty(\mu)$ with norm 1 by

$$\Lambda(f) = \int_{\mathbb{R}} f(t)\, M(t)\, g(t)^*\, d\tau\mu(t), \qquad \text{for every function} \quad f \in \mathscr{L}^1(\mu).$$

Since $\Lambda(p) = 0$ for any polynomial p we deduce that the measures in the entries of $M(t)\, g(t)^*\, d\tau\mu$ have all null moments, and one of these measures is not zero since on the contrary we would have $\Lambda \equiv 0$. We put $d(t)\, d\tau\mu$ for the real or imaginary non-zero part of one of these measures, and we suppose it is obtained when multiplying the i_0th row of $M(t)$ with the vector $g(t)^*$. Observe that

$$|d(t)| \leqslant |m_{i_0,1}(t)\, \overline{g_1(t)} + \cdots + m_{i_0,N}(t)\, \overline{g_N(t)}| \leqslant N$$

because $|m_{i,j}| \leqslant 1$ and $|g_i| \leqslant 1$, for $0 \leqslant i, j \leqslant N$. We decompose μ in the following way:

$$\mu = \frac{1}{2}\left\{\left(M(t) + \frac{\varepsilon}{2N} d(t)\, I\right) d\tau\mu + \left(M(t) - \frac{\varepsilon}{2N} d(t)\, I\right) d\tau\mu\right\}$$
$$= \frac{1}{2}(\mu_1 + \mu_2).$$

From the construction, it is clear that μ_1 and μ_2 both have the same moments as μ. Let us see now that these two matrices of measures are positive definite. For this it is enough to see that for every point t in the support of μ the numerical matrices $M(t) \pm (\varepsilon/2N)\, d(t)\, I$ are positive semi-definite. Given an arbitrary vector c in $\mathbb{C}^N$ we write it as $c = \sum_{i=1}^{N} c_i v_i$, where $v_1, ..., v_n$ is an orthonormal basis of $\mathbb{C}^N$, v_i associated to λ_i. We then have that

$$c\left(M(t) \pm \frac{\varepsilon}{2N} d(t)\, I\right) c^* = \sum_{i=1}^{N} |c_i|^2\, \lambda_i(t) \pm \frac{\varepsilon}{2N} d(t) \sum_{i=1}^{N} |c_i|^2$$
$$\geqslant \varepsilon \sum_{i=1}^{N} |c_i|^2 - \frac{\varepsilon}{2N} N \sum_{i=1}^{N} |c_i|^2$$
$$= \frac{\varepsilon}{2} \sum_{i=1}^{N} |c_i|^2 \geqslant 0$$

and hence both matrices of measures are positive definite.

To prove now that (2) implies (1), suppose that μ is not extremal in the set $[\mu]$ and let us prove that the polynomials are not dense in the space $\mathscr{L}^1(\mu)$. Suppose then that μ admits the decomposition $\mu = \frac{1}{2}(\mu_1 + \mu_2)$, where μ_1 and μ_2 are two matrices of measures in the set $[\mu]$, none of them equal to μ. Since $\mu_1 \leqslant 2\mu \leqslant 2\tau\mu I$ and $\mu_2 \leqslant 2\mu \leqslant 2\tau\mu I$, we can express $\mu_1 = M_1(t)\, d\tau\mu$ and $\mu_2 = M_2(t)\, d\tau\mu$, the matrices $M_1(t)$ and $M_2(t)$ being the Radon–Nikodym derivatives of the matrices μ_1 and μ_2 with respect to $\tau\mu$, respectively, that is,

$$\mu = M(t)\, d\tau\mu = \tfrac{1}{2}(M_1(t) + M_2(t))\, d\tau\mu.$$

Since μ is not equal to μ_1 we have that $M(t)$ is not equal to $M_1(t)$ and hence there exists a vector $e_{i_0} = (0, ..., 1, ..., 0)$, where the 1 is at the position i_0 such that $M(t)\, e_{i_0}^* \neq M_1(t)\, e_{i_0}^*$. We define the operators T and T_1 acting on $\mathscr{L}^1(\mu)$ by

$$T(f) = \int_{\mathbb{R}} f(t)\, M(t)\, e_{i_0}^*\, d\tau\mu \qquad \text{and} \qquad T_1(f) = \int_{\mathbb{R}} f(t)\, M_1(t)\, e_{i_0}^*\, d\tau\mu.$$

It is clear that both of them are linear and that T is continuous because e_{i_0} belongs to $\mathscr{L}^\infty(\mu)$. To show that T_1 is also continuous, observe first that from $M_1 \leqslant 2M \leqslant 2I$ we get $M_1^2 \leqslant 4I$. Moreover, since $\lambda_1 \geqslant \varepsilon$ we have that $M \geqslant \varepsilon I$ and so $M^2 \geqslant \varepsilon^2 I$. Hence if we call $C = 4/\varepsilon^2$ we have $CM^2 \geqslant 4I \geqslant M_1^2$. From this and from Lemma 2.2 we have

$$
\begin{aligned}
|T_1(f)| &= \left| \int_{\mathbb{R}} f(t)\, M_1(t)\, e_{i_0}^*\, d\tau\mu \right| \\
&\leqslant \int_{\mathbb{R}} |f(t)\, M_1(t)\, e_{i_0}^*|\, d\tau\mu \\
&\leqslant \int_{\mathbb{R}} (f(t)\, M_1^2(t)\, f(t)^*)^{1/2}\, (e_{i_0} M_1^2(t)\, e_{i_0}^*)^{1/2}\, d\tau\mu \\
&\leqslant \sqrt{2}\, C^{1/2} \int_{\mathbb{R}} (f(t)\, M_1^2(t)\, f(t)^*)^{1/2}\, d\tau\mu \\
&= \sqrt{2}\, C\, \|f\|_{1,\mu}.
\end{aligned}
$$

Then if we define $R = T - T_1$, R is a non-null operator acting on $\mathscr{L}^1(\mu)$ which vanishes on the polynomials; hence the polynomials are not dense in the space $\mathscr{L}^1(\mu)$, and the result is proved. ▮

As we proved in Section 3 the set $[\mu]$ is compact and convex. This guarantees the existence of extremal points. The general question, whether these extremal points are the same as those having dense polynomials in their corresponding spaces $\mathscr{L}^1(\mu)$, without conditions on the structure of the eigenvalues of their Random–Nykodym matrix of derivatives, is thus left as an open problem.

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